

NONEXISTENCE OF STABLE SOLUTIONS TO QUASILINEAR ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We prove nonexistence of nontrivial, possibly sign changing, stable solutions to a class of quasilinear elliptic equations with a potential on Riemannian manifolds, under suitable weighted growth conditions on geodesic balls.

1. INTRODUCTION

In this paper we investigate nonexistence of nontrivial, possibly sign changing, *stable* solutions of equation

$$(1) \quad -\Delta_p u = V(x)|u|^{\sigma-1}u \quad \text{in } M,$$

where M is a complete, non-compact Riemannian manifold of dimension m endowed with a metric tensor g , V is a given positive function, Δ_p denotes the p -Laplace operator, that is

$$\Delta_p u := \operatorname{div} \{ |\nabla u|^{p-2} \nabla u \},$$

where div and ∇ are the divergence operator and the gradient associated to the metric g , respectively. In particular, for $p = 2$, Δ_p becomes the standard Laplace-Beltrami operator on M . Concerning the parameters σ and p , and the potential V , we always assume that

$$\sigma > p - 1 \geq 1,$$

and that

$$V \in L^1_{\operatorname{loc}}(M), \quad V > 0 \quad \text{a.e. in } M.$$

The precise notions of solutions and of stability are given in Definitions 2.1 and 2.2 below, respectively.

Starting from the seminal paper [11], for $M = \mathbb{R}^m$, $V \equiv 1$ solutions of equation (1) have been largely studied in the literature (see, e.g., [1]-[14], [17], [20]-[23]). In particular, in [11] it is shown that if $M = \mathbb{R}^m$, $V \equiv 1$, $p = 2$ and

$$(2) \quad \begin{cases} 1 < \sigma < +\infty & \text{if } m \leq 10, \\ 1 < \sigma < \sigma_c(m) := \frac{(m-2)^2 - 4m + 8\sqrt{m-1}}{(m-2)(m-10)} & \text{if } m \geq 11, \end{cases}$$

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then the unique stable solution of equation (1) is $u \equiv 0$. Furthermore, the previous result has been generalized to the case $p > 2$ in [6]. Indeed, when $M = \mathbb{R}^m$, $V \equiv 1$, $p > 2$, it is established that if u is a stable solution of equation (1) and

$$(3) \quad \begin{cases} p-1 < \sigma < +\infty & \text{if } m \leq \frac{p(p+3)}{p-1}, \\ p-1 < \sigma < \sigma_c(m, p) & \text{if } m > \frac{p(p+3)}{p-1}, \end{cases}$$

with

$$(4) \quad \sigma_c(m, p) := \frac{[(p-1)m - p]^2 + p^2(p-2) - p^2(p-1)m + 2p^2\sqrt{(p-1)(m-1)}}{(m-p)[(p-1)m - p(p+3)]},$$

then $u \equiv 0$ in \mathbb{R}^m . We should observe that in [11] and in [6] more regular solutions are used; specifically, solutions of class C^2 and $C_{\text{loc}}^{1,\alpha}$, respectively, are considered. However, in view of standard regularity results (see [7]), weak solutions meant in the sense of Definition 2.1 belong to those classes, e.g. if V is locally bounded.

On the other hand, nonexistence of *positive* solutions (not necessarily stable) of equation (1) has also been studied on Riemannian manifold (see, e.g., [15], [16], [18], [26], [27]). More precisely, denote by $d\mu$ the canonical Riemannian volume element on M ; fix any reference point $o \in M$ and set $B_R := \{x \in M : \text{dist}(x, o) < R\}$. Furthermore, let

$$(5) \quad \alpha_0 := \frac{p\sigma}{\sigma - p + 1}, \quad \beta_0 := \frac{p-1}{\sigma - p + 1}.$$

In [18] it is proved that equation (1) does not admit any nontrivial positive solution, provided that there exist $C_0 > 0$, $k \in (0, \beta_0)$ such that, for every $R > 0$ sufficiently large and for every $\varepsilon > 0$ sufficiently small,

$$(6) \quad \int_{B_R \setminus B_{\frac{R}{2}}} V^{-\beta_0 + \varepsilon} d\mu \leq CR^{\alpha_0 + C_0\varepsilon} (\log R)^k.$$

The same conclusion remains true, if instead of (6) it is assumed that there exist $C_0 > 0$ such that, for every $R > 0$ sufficiently large and for every $\varepsilon > 0$ sufficiently small,

$$\int_{B_R \setminus B_{\frac{R}{2}}} V^{-\beta_0 + \varepsilon} d\mu \leq CR^{\alpha_0 + C_0\varepsilon} (\log R)^{\beta_0},$$

and

$$\int_{B_R \setminus B_{\frac{R}{2}}} V^{-\beta_0 - \varepsilon} d\mu \leq CR^{\alpha_0 + C_0\varepsilon} (\log R)^{\beta_0}.$$

Such results are in agreement with those in [23] for $M = \mathbb{R}^m$. In fact, in this case no nonnegative, nontrivial solution exists, provided that

$$V \equiv 1, \quad m > p, \quad p-1 < \sigma < \frac{m(p-1)}{m-p}.$$

Similar results have been also obtained for parabolic equations (see, e.g., [19], [23], [24]).

The aim of this paper is to study nonexistence of *stable*, possibly sign changing solutions of equation (1) on Riemannian manifolds. We show that no nontrivial stable solution exists, provided that a condition similar to (6) is satisfied (see Theorems 2.5 and 2.6 below). Note that for $M = \mathbb{R}^m$, $V \equiv 1$, our results are in accordance with those stated in [6] (see Example 2.8 below). In order to prove our results, first we deduce suitable a priori estimates for stable solutions of equation (1), by using the definition of stable weak solutions (see Propositions

3.1 and 3.2 below). Then by choosing an appropriate family of test functions, depending on two parameters, in such a priori estimates, we infer that any stable solution u of equation (1) vanishes identically.

The estimate in Proposition 3.1 is similar to that in [6, Proposition 1.4] (where $V \equiv 1$, $M = \mathbb{R}^m$); however, it is more accurate. In particular, we explicitly compute the presence of $\delta = \bar{\gamma} - \gamma$ in the r.h.s. of the integral estimate (19), see also (8) and (9); this will be expedient in order to use the appropriate test functions in the sequel. On the other hand, the estimate in Proposition 3.2 is new.

Let us mention that we do not use the same test functions as in [6]. Indeed, by doing so, weaker results than ours will be obtained. We use the same family of test functions, depending on two parameters, employed in [16], [18]. However, in [16], [18] such test functions were used in different a priori estimates, therefore in our situation in order to get the conclusion new estimates are necessary.

The paper is organized as follows. In Section 2 we state our main results, which are then proved in Section 4. Section 3 contains some important auxiliary tools, that are used in the proofs of the main theorems.

2. STATEMENT OF THE MAIN RESULTS

Denote by $\text{Lip}_{\text{loc}}(M)$ the set of Lipschitz functions $\varphi : M \rightarrow \mathbb{R}$ having compact support. For any open domain $\Omega \subset M$ and $p > 1$, let $W^{1,p}(\Omega)$ be the completion with respect to the norm

$$\|w\|_{W^{1,p}(\Omega)} = \left(\int_{\Omega} |w|^p d\mu + \int_{\Omega} |\nabla w|^p d\mu \right)^{\frac{1}{p}}$$

of the space of locally Lipschitz functions $w : \Omega \rightarrow \mathbb{R}$ having finite $W^{1,p}(\Omega)$ norm. For any function $u : M \rightarrow \mathbb{R}$, we say that $u \in W_{\text{loc}}^{1,p}(M)$ if for every $\varphi \in \text{Lip}_{\text{loc}}(M)$ one has that $u\varphi \in W^{1,p}(M)$.

Definition 2.1. *We say that $u \in W_{\text{loc}}^{1,p}(M) \cap L_{\text{loc}}^{\sigma}(M)$ is a weak solution of (1) if, for every $\varphi \in \text{Lip}_{\text{loc}}(M)$, one has*

$$(7) \quad \int_M |\nabla u|^{p-2} (\nabla u, \nabla \varphi) d\mu = \int_M V(x) |u|^{\sigma-1} u \varphi d\mu,$$

where (\cdot, \cdot) is the scalar product associated to the metric g .

We explicitly note that the solution u in the above definition can change sign and can be unbounded on M . Moreover, it is well known that the class of test functions used in Definition 2.1 can be equivalently restricted to $\varphi \in C_c^{\infty}(M)$.

The *linearized operator* of (7) at u is given by

$$\begin{aligned} L_u(v, \varphi) &= \int_M |\nabla u|^{p-2} (\nabla v, \nabla \varphi) d\mu + \\ &+ \int_M \{ (p-2) |\nabla u|^{p-4} (\nabla u, \nabla v) g(\nabla u, \nabla \varphi) - \sigma V(x) |u|^{\sigma-1} v \varphi \} d\mu \end{aligned}$$

for every $v, \varphi \in \text{Lip}_{\text{loc}}(M)$.

Definition 2.2. *We say that a weak solution $u \in W_{\text{loc}}^{1,p}(M) \cap L_{\text{loc}}^{\sigma}(M)$ of equation (1) is stable, if*

$$L_u(\varphi, \varphi) \geq 0 \quad \forall \varphi \in \text{Lip}_{\text{loc}}(M).$$

To state the following result we need to introduce some notation. We set

$$(8) \quad \bar{\gamma} = \bar{\gamma}_{p,\sigma} := \frac{2\sigma - p + 1 + 2\sqrt{\sigma(\sigma - p + 1)}}{p - 1}$$

and, for any $\gamma \in [\frac{\sigma}{p-1}, \bar{\gamma})$ we set

$$(9) \quad \delta = \delta_\gamma := \bar{\gamma} - \gamma.$$

Let $r(x)$ denote the geodesic distance of a point $x \in M$ from a fixed origin $o \in M$ and for every $R > 0$ let B_R be the geodesic ball with radius o and radius R .

Definition 2.3. *Let*

$$(10) \quad \alpha := p \frac{\sigma + \bar{\gamma}}{\sigma - p + 1} \quad \beta := \frac{\bar{\gamma} + p - 1}{\sigma - p + 1}.$$

We say that condition (HP1) holds if there exist positive constants C, C_0, R_0, ϵ_0 such that for every $\epsilon \in (0, \epsilon_0)$ and $R \in (R_0, \infty)$ one has

$$(11) \quad \int_{B_R \setminus B_{R/2}} V^{-\beta+\epsilon} d\mu \leq CR^{\alpha+C_0\epsilon} (\log R)^{-1}.$$

We say that condition (HP2) holds if there exist positive constants $C, C_0, R_0, \epsilon_0, \theta$ and a constant $b < -1 + \frac{\theta}{\sigma-p+1}$ such that for every $\epsilon \in (0, \epsilon_0)$ and $R \in (R_0, \infty)$ one has

$$(12) \quad \int_{B_R \setminus B_{R/2}} V^{-\beta+\epsilon} d\mu \leq CR^{\alpha+C_0\epsilon} (\log R)^b e^{-\theta\epsilon \log \frac{R}{2} \log \log \frac{R}{2}}.$$

Let us observe that

$$\alpha > \alpha_0, \quad \beta > \beta_0,$$

where α_0, β_0 are defined in (5).

Remark 2.4. *We note that condition (HP1) holds if there exist positive constants C, C_0 such that*

$$(13) \quad 0 < V(x) \leq C(1 + r(x))^{C_0} \quad \text{in } M$$

and

$$(14) \quad \int_{B_R \setminus B_{R/2}} V^{-\beta} d\mu \leq CR^\alpha (\log R)^{-1}$$

for every $R > 0$ sufficiently large.

Similarly, condition (HP2) holds if for some positive constants C, C_0, θ one has

$$(15) \quad 0 < V(x) \leq Cr(x)^{C_0} e^{-\theta \log r(x) \log \log r(x)}$$

for every $x \in M \setminus B_{R^*}$ for some fixed $R^* > 0$ and

$$(16) \quad \int_{B_R \setminus B_{R/2}} V^{-\beta} d\mu \leq CR^\alpha (\log R)^b$$

for some $b < -1 + \frac{\theta}{\sigma-p+1}$ and every $R > 0$ sufficiently large.

Now we can state our main nonexistence results.

Theorem 2.5. *Suppose that condition (HP1) holds. Let u be a stable solution of equation (1), then $u \equiv 0$ in M .*

Theorem 2.6. *Suppose that condition (HP2) holds. Let u be a stable solution of equation (1), then $u \equiv 0$ in M .*

By Theorem 2.6 we immediately have the following corollary.

Corollary 2.7. *If for some positive constants C , C_0 , θ one has*

$$(17) \quad 0 < V(x) \leq C(1 + r(x))^{C_0} e^{-\theta r(x)} \quad \text{in } M$$

and (16) for some $b \in \mathbb{R}$ and every $R > 0$ large enough. Let u be a stable solution of equation (1). Then $u \equiv 0$ in M .

Example 2.8. In the case $M = \mathbb{R}^m$ and $V \equiv 1$, condition (HP1) is satisfied if and only if

$$(18) \quad m < \alpha = \alpha(\sigma) = p \frac{\sigma + \bar{\gamma}(\sigma)}{\sigma - p + 1},$$

with $\bar{\gamma}(\sigma)$ defined in (8). It is easy to see that $\alpha(\sigma)$ is a strictly decreasing function for $\sigma \in (p - 1, \infty)$, with

$$\lim_{\sigma \rightarrow (p-1)^+} \alpha(\sigma) = +\infty, \quad \lim_{\sigma \rightarrow +\infty} \alpha(\sigma) = \frac{p(p+3)}{p-1}.$$

Therefore, if $m \leq \frac{p(p+3)}{p-1}$, condition (18) is satisfied for every $\sigma > p - 1$. On the other hand, if $m > \frac{p(p+3)}{p-1}$, there exists a unique $\sigma^* \in (p - 1, \infty)$ such that $\alpha(\sigma^*) = m$ and condition (18) is satisfied if and only if $\sigma \in (p - 1, \sigma^*)$. An easy computation shows that

$$\sigma^* = \sigma_c(m, p),$$

with $\sigma_c(m, p)$ defined in (4). Hence if $V \equiv 1$ and (3) hold, from Theorem 2.5 it follows that every weak solution u of equation (1) which is stable in \mathbb{R}^n vanishes identically. Note that the result is in accordance with [6, Theorem 1.5] for $p > 2$ and with [11, Theorem 1] for $p = 2$. Moreover, for $p = 2$ in [11, Theorem 1] it is shown that the result is sharp, in the sense that if the requirement on the parameter σ is not fulfilled, then a stable solution exists.

Remark 2.9. *Example 2.8 shows that the exponent α introduced in (10) is sharp in condition (HP1), in order to obtain nonexistence of nontrivial stable solutions of (1). In particular, see Remark 2.4, the exponent α on the radius R is sharp in the weighted volume growth assumption (14), for the class of potential functions V satisfying (13).*

On the other hand, Theorem 2.6 and Corollary 2.7 show that the exponent -1 on the logarithm of the radius R in the weighted volume growth condition (14) is not sharp in general and can be increased, if one strengthens the assumptions on the potential V , in particular imposing hypotheses on its behavior at infinity such as (15) or (17).

3. AUXILIARY RESULTS

In this Section we prove the following two propositions, that will have a crucial role in the proof of Theorems 2.5 and 2.6.

Proposition 3.1. *Let $u \in W_{\text{loc}}^{1,p}(M) \cap L_{\text{loc}}^\sigma(M)$ be a stable solution of (7) with $\sigma > (p - 1)$ and $p \geq 2$. Let also $\bar{\gamma}$, δ_γ , γ be as in (8) and (9) and consider k such that*

$$k \geq \max \left\{ \frac{p + \gamma}{\sigma - p + 1}, 2 \right\}.$$

Then there exists a positive constant $c = c(p, \sigma, k)$ such that

$$(19) \quad \int_M V(x) |u|^{\sigma+\gamma} \psi^{pk} d\mu \leq c \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} \int_M V(x)^{-\frac{\gamma+p-1}{\sigma-p+1}} |\nabla \psi|^p \frac{\sigma+\gamma}{\sigma-p+1} d\mu$$

and

$$(20) \quad \int_M |\nabla u|^p |u|^{\gamma-1} \psi^{pk} d\mu \leq c \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} \int_M V(x)^{-\frac{\gamma+p-1}{\sigma-p+1}} |\nabla \psi|^p \frac{\sigma+\gamma}{\sigma-p+1} d\mu$$

for all test functions $\psi \in \text{Lip}_c(M)$ with $0 \leq \psi \leq 1$.

Proposition 3.2. *Under the same assumptions of Proposition 3.1, for every $\delta > 0$ small enough one has*

$$(21) \quad \int_M V(x) |u|^{\sigma+\gamma} \psi^{pk} d\mu \leq c \left[\int_{M \setminus K} V(x) |u|^{\sigma+\gamma} d\mu \right]^{\frac{\gamma+p-1}{\sigma+\gamma}} \left[\delta^{-p \frac{\sigma+\gamma}{\sigma-p+1+\delta}} \int_{M \setminus K} V(x)^{-\frac{\gamma+p-1}{\sigma-p+1+\delta}} |\nabla \psi|^p \frac{\sigma+\gamma}{\sigma-p+1+\delta} d\mu \right]^{\frac{\sigma-p+1+\delta}{\sigma+\gamma}}$$

for some positive constant c , where $K = \{x \in M \mid \psi(x) = 1\}$.

Proposition 3.1 provides an important estimate on the integrability of u and ∇u . As we will see, our nonexistence results will follow by showing that the right-hand sides vanish, under suitably weighted volume growth assumptions on geodesic balls or annuli.

Proof of Proposition 3.1. Step 1. For any non-negative $\varphi \in \text{Lip}_c(\Omega)$, by density arguments, we can plug

$$\Phi = |u|^{\gamma-1} u \varphi^p$$

in (7) and get

$$\gamma \int_M |\nabla u|^p |u|^{\gamma-1} \varphi^p d\mu = -p \int_M |\nabla u|^{p-2} (\nabla u, \nabla \varphi) |u|^{\gamma-1} u \varphi^{p-1} d\mu + \int_M V(x) |u|^{\sigma+\gamma} \varphi^p d\mu$$

so that

$$\gamma \int_M |\nabla u|^p |u|^{\gamma-1} \varphi^p d\mu \leq p \int_M |\nabla u|^{p-1} |\nabla \varphi| |u|^\gamma \varphi^{p-1} d\mu + \int_M V(x) |u|^{\sigma+\gamma} \varphi^p d\mu.$$

Writing $|u|^\gamma = |u|^{[(\gamma-1)\frac{p-1}{p} + \frac{\gamma+(p-1)}{p}]}$ and exploiting Young's inequality with exponents p and $\frac{p}{p-1}$ we have

$$\begin{aligned} \gamma \int_M |\nabla u|^p |u|^{\gamma-1} \varphi^p d\mu &\leq \epsilon \int_M |\nabla u|^p |u|^{\gamma-1} \varphi^p d\mu \\ &\quad + \frac{c(p)}{\epsilon^{p-1}} \int_M |\nabla \varphi|^p |u|^{\gamma+p-1} d\mu + \int_M V(x) |u|^{\sigma+\gamma} \varphi^p d\mu \end{aligned}$$

that we rewrite as

$$(22) \quad (\gamma - \epsilon) \int_M |\nabla u|^p |u|^{\gamma-1} \varphi^p d\mu \leq \frac{c(p)}{\epsilon^{p-1}} \int_M |\nabla \varphi|^p |u|^{\gamma+(p-1)} d\mu + \int_M V(x) |u|^{\sigma+\gamma} \varphi^p d\mu.$$

Step 2. Now we exploit the stability of u with the choice of test function

$$\tilde{\Phi} = |u|^{\frac{\gamma-1}{2}} u \varphi^{\frac{p}{2}}$$

and we get

$$(23) \quad \begin{aligned} \frac{\sigma}{p-1} \int_M V(x) |u|^{\sigma+\gamma} \varphi^p &\leq \left(\frac{1+\gamma}{2} \right)^2 \int_M |\nabla u|^p |u|^{\gamma-1} \varphi^p + \frac{p^2}{4} \int_M |\nabla u|^{p-2} |u|^{\gamma+1} |\nabla \varphi|^2 \varphi^{p-2} \\ &\quad + p \frac{\gamma+1}{2} \int_M |\nabla u|^{p-1} |u|^\gamma \varphi^{p-1} |\nabla \varphi|. \end{aligned}$$

In case $p > 2$ we write $|u|^{\gamma+1} = |u|^{[(\gamma-1)\frac{(p-2)}{p} + 2\frac{(\gamma+(p-1))}{p}]}$ and using Young's inequality with exponents $\frac{p}{p-2}$ and $\frac{p}{2}$ we get

$$(24) \quad \frac{p^2}{4} \int_M |\nabla u|^{p-2} |u|^{\gamma+1} \varphi^{p-2} |\nabla \varphi|^2 \leq \frac{\epsilon}{2} \int_M |\nabla u|^p |u|^{\gamma-1} \varphi^p + \frac{c(p)}{\epsilon^{\frac{p-2}{2}}} \int_M |\nabla \varphi|^p |u|^{\gamma+(p-1)}.$$

Also, by writing $|u|^\gamma = |u|^{(\gamma-1)\frac{p-1}{p} + \frac{\gamma+(p-1)}{p}}$ and using Young's inequality with exponents $\frac{p}{p-1}$ and p , we obtain

$$(25) \quad p \frac{\gamma+1}{2} \int_M |\nabla u|^{p-1} |u|^\gamma \varphi^{p-1} |\nabla \varphi| \leq \frac{\epsilon}{2} \int_M |\nabla u|^p \varphi^p |u|^{\gamma-1} + \frac{c(p)}{\epsilon^{p-1}} \int_M |\nabla \varphi|^p |u|^{\gamma+(p-1)}.$$

Recollecting and exploiting (23), we get

$$(26) \quad \begin{aligned} \frac{\sigma}{p-1} \int_M V(x) |u|^{\sigma+\gamma} \varphi^p &\leq \left[\left(\frac{\gamma+1}{2} \right)^2 + \epsilon \right] \int_M |\nabla u|^p |u|^{\gamma-1} \varphi^p + \\ &\quad + c(p) \left(\frac{1}{\epsilon^{\frac{p-2}{2}}} + \frac{1}{\epsilon^{p-1}} \right) \int_M |\nabla \varphi|^p |u|^{\gamma+(p-1)}. \end{aligned}$$

By (22) and (26),

$$(27) \quad \begin{aligned} \left(\frac{\sigma}{p-1} \right) \int_M V(x) |u|^{\sigma+\gamma} \varphi^p &\leq \left[\left(\frac{\gamma+1}{2} \right)^2 + \epsilon \right] \frac{1}{(\gamma-\epsilon)} \int_M V(x) |u|^{\sigma+\gamma} \varphi^p \\ &\quad + c(p, \gamma) \left(\frac{1}{\epsilon^{\frac{p-2}{2}}} + \frac{1}{\epsilon^{p-1}} \right) \int_M |\nabla \varphi|^p |u|^{\gamma+(p-1)}. \end{aligned}$$

In case $p = 2$ one does not need inequality (24). Indeed, starting from (23) and using (22) and (25) one can easily see that (27) holds also when $p = 2$.

Since $c(p, \gamma)$ is bounded away from zero and from infinity in the range of γ that we are considering, from now on we omit the dependence on γ . Setting

$$(28) \quad \omega = \omega_\epsilon = \frac{\sigma}{p-1} - \left(\frac{(\gamma+1)^2}{4} + \epsilon \right) \frac{1}{\gamma-\epsilon}$$

and considering with no loss of generality bounded values of ϵ , e.g. $0 < \epsilon \leq 2^{-1}$, we arrive at

$$(29) \quad \omega \int_M V(x) |u|^{\sigma+\gamma} \varphi^p \leq \frac{c(p)}{\epsilon^{p-1}} \int_M |\nabla \varphi|^p |u|^{\gamma+(p-1)}$$

for any non-negative $\varphi \in \text{Lip}_c(M)$. It is now convenient to set

$$g(\gamma) := \frac{\sigma}{p-1} - \frac{(\gamma+1)^2}{4\gamma}.$$

By direct computation it follows that $g(\bar{\gamma}) = 0$ and $g'(\bar{\gamma}) < 0$ for $\gamma > 1$. Therefore, for the range of δ we are considering, recalling that $\delta = \bar{\gamma} - \gamma$, we infer that $g'(\gamma)$ is bounded away from zero. Now, by the mean value theorem, we deduce that

$$\begin{aligned}\omega_\epsilon &= \frac{\sigma}{p-1} - \left(\frac{(\gamma+1)^2}{4} + \epsilon \right) \frac{1}{\gamma - \epsilon} \\ &= -(g(\bar{\gamma}) - g(\gamma)) - \frac{(\gamma+1)^2}{4} \cdot \frac{\epsilon}{(\gamma - \epsilon)\gamma} - \frac{\epsilon}{(\gamma - \epsilon)} \\ &\geq 2\tilde{c}(p, \sigma)\delta - \epsilon \left(\frac{(\gamma+1)^2}{2\gamma} + 2 \right),\end{aligned}$$

where we exploited the fact that we are assuming e.g. $0 < \epsilon \leq 2^{-1}$ and by assumption $\gamma > 1$, so that $\gamma - \epsilon \geq 2^{-1}$. Taking into account (8), we deduce that

$$\omega_\epsilon > \tilde{c}(p, \sigma)\delta \quad \text{for} \quad \epsilon = \tilde{c}(p, \sigma) \cdot \left(\frac{(\gamma+1)^2}{2\gamma} + 2 \right)^{-1} \delta.$$

Therefore (29) can be rewritten as

$$(30) \quad \int_M V(x) |u|^{\sigma+\gamma} \varphi^p \leq \frac{c(p, \sigma)}{\delta^p} \int_M |\nabla \varphi|^p |u|^{\gamma+(p-1)}.$$

Furthermore, possibly relabeling the constant, (22) provides

$$(31) \quad \int_M |\nabla u|^p |u|^{\gamma-1} \varphi^p \leq \frac{c(p, \sigma)}{\delta^p} \int_M |\nabla \varphi|^p |u|^{\gamma+(p-1)}.$$

Step 3. Let us now consider

$$\psi \in \text{Lip}_c(M)$$

with $0 \leq \psi \leq 1$ and take $\varphi = \psi^k$ in (30). Then we obtain

$$(32) \quad \int_M V(x) |u|^{\sigma+\gamma} \psi^{pk} d\mu \leq \frac{c(p, \sigma, k)}{\delta^p} \int_M |u|^{\gamma+(p-1)} \psi^{p(k-1)} |\nabla \psi|^p d\mu.$$

Hence

$$\begin{aligned}\int_M V(x) |u|^{\sigma+\gamma} \psi^{pk} d\mu &\leq \frac{c(p, \sigma, k)}{\delta^p} \left[\int_M \left(V(x)^{\frac{\gamma+(p-1)}{\sigma+\gamma}} |u|^{\gamma+(p-1)} \psi^{p(k-1)} \right)^{\frac{\sigma+\gamma}{\gamma+(p-1)}} d\mu \right]^{\frac{\gamma+(p-1)}{\sigma+\gamma}} \\ &\quad \times \left[\int_M \left(V(x)^{-\frac{\gamma+(p-1)}{\sigma+\gamma}} |\nabla \psi|^p \right)^{\frac{\sigma+\gamma}{\sigma-(p-1)}} d\mu \right]^{\frac{\sigma-(p-1)}{\sigma+\gamma}},\end{aligned}$$

where we used the fact that $\sigma > (p-1)$. Furthermore

$$(33) \quad p(k-1) \left(\frac{\sigma+\gamma}{\gamma+(p-1)} \right) \geq pk$$

since we assumed $k \geq \frac{\sigma+\gamma}{\sigma-(p-1)}$. Consequently, recalling that $0 \leq \psi \leq 1$, we have

$$\begin{aligned}\int_M V(x) |u|^{\sigma+\gamma} \psi^{pk} &\leq \\ &\frac{c(p, \sigma, k)}{\delta^p} \left[\int_M V(x) |u|^{\sigma+\gamma} \psi^{pk} \right]^{\frac{\gamma+(p-1)}{\sigma+\gamma}} \left[\int_M V(x)^{-\frac{\gamma+(p-1)}{\sigma-(p-1)}} |\nabla \psi|^p \right]^{\frac{\sigma-(p-1)}{\sigma+\gamma}}.\end{aligned}$$

Hence

$$\int_M V(x) |u|^{\sigma+\gamma} \psi^{pk} \leq c(p, \sigma, k) \delta^{-p \frac{\sigma+\gamma}{\sigma-p+1}} \int_M V(x)^{-\frac{\gamma+(p-1)}{\sigma-(p-1)}} |\nabla \psi|^p \frac{\sigma+\gamma}{\sigma-(p-1)}.$$

that is (19).

Step 4. Again we consider $\psi \in \text{Lip}_c(M)$ such that $0 \leq \psi \leq 1$. Then we evaluate (31) for $\varphi = \psi^k$, obtaining

$$\begin{aligned} \int_M |\nabla u|^p |u|^{\gamma-1} \psi^{pk} &\leq \frac{c(p, \sigma, k)}{\delta^p} \int_M |u|^{\gamma+(p-1)} \psi^{p(k-1)} |\nabla \psi|^p \\ &\leq \frac{c(p, \sigma, k)}{\delta^p} \left[\int_M \left(V(x)^{\frac{\gamma+(p-1)}{\sigma+\gamma}} |u|^{\gamma+(p-1)} \psi^{p(k-1)} \right)^{\frac{\sigma+\gamma}{\gamma+(p-1)}} \right]^{\frac{\gamma+(p-1)}{\sigma+\gamma}} \\ &\quad \times \left[\int_M \left(V(x)^{-\frac{\gamma+(p-1)}{\sigma+\gamma}} |\nabla \psi|^p \right)^{\frac{\sigma+\gamma}{\sigma-(p-1)}} \right]^{\frac{\sigma-(p-1)}{\sigma+\gamma}} \\ &\leq \frac{c(p, \sigma, k)}{\delta^p} \left[\int_M (V(x) |u|^{\sigma+\gamma} \psi^{pk}) \right]^{\frac{\gamma+(p-1)}{\sigma+\gamma}} \left[\int_M \left(V(x)^{-\frac{\gamma+(p-1)}{\sigma+\gamma}} |\nabla \psi|^p \right)^{\frac{\sigma+\gamma}{\sigma-(p-1)}} \right]^{\frac{\sigma-(p-1)}{\sigma+\gamma}} \end{aligned}$$

and so, taking into account (19), we get (20), namely

$$\int_M |\nabla u|^p |u|^{\gamma-1} \psi^{pk} \leq \frac{c(p, \sigma, k)}{\delta^p} \left[\frac{c(p, \sigma, k)}{\delta^p \frac{\sigma+\gamma}{\sigma-p+1}} \right]^{\frac{\gamma+(p-1)}{\sigma+\gamma}} \int_M V(x)^{-\frac{\gamma+(p-1)}{\sigma-(p-1)}} |\nabla \psi|^p \frac{\sigma+\gamma}{\sigma-(p-1)}.$$

□

Proof of Proposition 3.2. Note that, since we are under the same assumptions of Proposition 3.1, (32) holds. Since $\nabla \psi = 0$ a.e. in K we have

$$\int_M V(x) |u|^{\sigma+\gamma} \psi^{pk} d\mu \leq \frac{c}{\delta^p} \int_{M \setminus K} |u|^{\gamma+(p-1)} \psi^{p(k-1)} |\nabla \psi|^p d\mu.$$

We now apply Hölder's inequality to obtain

$$\begin{aligned} \int_M V(x) |u|^{\sigma+\gamma} \psi^{pk} d\mu &\leq \frac{c}{\delta^p} \left[\int_{M \setminus K} V |u|^{\sigma+\gamma} \psi^{p(k-1)} \frac{\sigma+\gamma}{\gamma+p-1} d\mu \right]^{\frac{\gamma+p-1}{\sigma+\gamma}} \left[\int_{M \setminus K} |\nabla \psi|^p \frac{\sigma+\gamma}{\sigma-p+1+\delta} V^{-\frac{\gamma+p-1}{\sigma-p+1+\delta}} d\mu \right]^{\frac{\sigma-p+1+\delta}{\sigma+\gamma}} \\ &\leq c \left[\int_{M \setminus K} V |u|^{\sigma+\gamma} d\mu \right]^{\frac{\gamma+p-1}{\sigma+\gamma}} \left[\delta^{-p \frac{\sigma+\gamma}{\sigma-p+1+\delta}} \int_{M \setminus K} |\nabla \psi|^p \frac{\sigma+\gamma}{\sigma-p+1+\delta} V^{-\frac{\gamma+p-1}{\sigma-p+1+\delta}} d\mu \right]^{\frac{\sigma-p+1+\delta}{\sigma+\gamma}}, \end{aligned}$$

since $0 \leq \psi \leq 1$, that is the conclusion. □

4. PROOF OF THEOREMS 2.5 AND 2.6

Proof of Theorem 2.5. We start by considering the case when there exist positive constants C, C_0 such that for every small enough $\epsilon > 0$ and every large enough $R > 0$ one has

$$(34) \quad \int_{B_R \setminus B_{R/2}} V^{-\beta+\epsilon} d\mu \leq CR^{\alpha+C_0\epsilon} (\log R)^b,$$

for some $b < -1$.

We use inequality (19) with the particular choice of test function

$$(35) \quad \psi(x) \equiv \psi_n(x) = \varphi(x)\eta_n(x),$$

where

$$(36) \quad \varphi(x) = \begin{cases} 1 & \text{for } r(x) < R, \\ \left(\frac{r(x)}{R}\right)^{-C_1\delta} & \text{for } r(x) \geq R, \end{cases}$$

with $R > 0$ fixed, $C_1 > 0$ a fixed constant to be chosen later large enough and $\delta = \bar{\gamma} - \gamma > 0$ as defined in (8) and (9), and for $n \in \mathbb{N}$

$$(37) \quad \eta_n(x) = \begin{cases} 1 & \text{for } r(x) < nR, \\ 2 - \frac{r(x)}{nR} & \text{for } nR \leq r(x) \leq 2nR, \\ 0 & \text{for } r(x) \geq 2nR. \end{cases}$$

We further specialize the choice of δ , and hence of γ , in inequality (19) by setting $\delta = \frac{1}{\log R}$. Here and for the rest of the proof we always assume that $R > 0$ is chosen large enough so that condition (11) holds.

Note that $\psi_n \in Lip_c(M)$ with $0 \leq \psi_n \leq 1$ on M and one has

$$\nabla \psi_n = \varphi \nabla \eta_n + \eta_n \nabla \varphi.$$

Hence for every $a > 0$ one has

$$|\nabla \psi_n|^a \leq 2^a (\varphi^a |\nabla \eta_n|^a + \eta_n^a |\nabla \varphi|^a).$$

Substituting in (19) one obtains

$$(38) \quad \int_M V |u|^{\sigma+\gamma} \psi_n^{pk} d\mu \leq C \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} [I_1 + I_2],$$

where

$$(39) \quad \begin{aligned} I_1 &= \int_M V^{-\frac{\gamma+p-1}{\sigma-p+1}} \varphi^{p \frac{\sigma+\gamma}{\sigma-p+1}} |\nabla \eta_n|^{p \frac{\sigma+\gamma}{\sigma-p+1}} d\mu, \\ I_2 &= \int_M V^{-\frac{\gamma+p-1}{\sigma-p+1}} \eta_n^{p \frac{\sigma+\gamma}{\sigma-p+1}} |\nabla \varphi|^{p \frac{\sigma+\gamma}{\sigma-p+1}} d\mu. \end{aligned}$$

Now we easily see that

$$(40) \quad \begin{aligned} \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} I_1 &= \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} \int_M V^{-\frac{\gamma+p-1}{\sigma-p+1}} \varphi^{p \frac{\sigma+\gamma}{\sigma-p+1}} |\nabla \eta_n|^{p \frac{\sigma+\gamma}{\sigma-p+1}} d\mu, \\ &= \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} \int_{B_{2nR} \setminus B_{nR}} V^{-\frac{\bar{\gamma}+p-1-\delta}{\sigma-p+1}} \varphi^{p \frac{\sigma+\gamma}{\sigma-p+1}} |\nabla \eta_n|^{p \frac{\sigma+\gamma}{\sigma-p+1}} d\mu, \\ &\leq \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} n^{-pC_1\delta \frac{\sigma+\gamma}{\sigma-p+1}} (nR)^{-p \frac{\sigma+\gamma}{\sigma-p+1}} \int_{B_{2nR} \setminus B_{nR}} V^{-\frac{\bar{\gamma}+p-1-\delta}{\sigma-p+1}} d\mu. \end{aligned}$$

Hence, for $R > 0$ large enough and thus $\delta > 0$ small enough, by condition (34) we have

$$\delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} I_1 \leq C \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} n^{-pC_1\delta \frac{\sigma+\gamma}{\sigma-p+1}} (nR)^{-p \frac{\sigma+\gamma}{\sigma-p+1}} (2nR)^{\alpha + \frac{C_0\delta}{\sigma-p+1}} (\log(2nR))^b.$$

By our definition of α , the previous inequality yields

$$(41) \quad \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} I_1 \leq C \delta^{-p \frac{\bar{\gamma}+\sigma-\delta}{\sigma-p+1}} n^{\frac{\delta}{\sigma-p+1} (C_0+p-pC_1(\sigma+\bar{\gamma}-\delta))} (R^\delta)^{\frac{p+C_0}{\sigma-p+1}} (\log(2nR))^b.$$

We choose $R > 0$ so large that $\delta = \frac{1}{\log R} < \frac{\sigma + \bar{\gamma}}{2}$ and

$$(42) \quad C_1 > 2 \frac{C_0 + p + 1}{p(\sigma + \bar{\gamma})}$$

in (36). Since $R^\delta = e$, from (41) we deduce

$$(43) \quad \delta^{-p \frac{\gamma + \sigma}{\sigma - p + 1}} I_1 \leq C \delta^{-p \frac{\bar{\gamma} + \sigma - \delta}{\sigma - p + 1}} n^{-\frac{\delta}{\sigma - p + 1}} (\log(2nR))^b.$$

Now we estimate

$$(44) \quad \begin{aligned} \delta^{-p \frac{\gamma + \sigma}{\sigma - p + 1}} I_2 &= \delta^{-p \frac{\gamma + \sigma}{\sigma - p + 1}} \int_M V^{-\frac{\gamma + p - 1}{\sigma - p + 1}} \eta_n^{p \frac{\sigma + \gamma}{\sigma - p + 1}} |\nabla \varphi|^{p \frac{\sigma + \gamma}{\sigma - p + 1}} d\mu \\ &\leq \delta^{-p \frac{\gamma + \sigma}{\sigma - p + 1}} \int_{M \setminus B_R} V^{-\frac{\gamma + p - 1}{\sigma - p + 1}} (C_1 \delta R^{C_1 \delta} r(x)^{-C_1 \delta - 1})^{p \frac{\sigma + \gamma}{\sigma - p + 1}} d\mu \\ &\leq C R^{C_1 p \delta \frac{\sigma + \gamma}{\sigma - p + 1}} \int_{M \setminus B_R} V^{-\frac{\bar{\gamma} + p - 1 - \delta}{\sigma - p + 1}} r(x)^{-(C_1 \delta + 1)p \frac{\sigma + \gamma}{\sigma - p + 1}} d\mu. \end{aligned}$$

In order to proceed with our estimates we recall that if $f : [0, \infty) \rightarrow [0, \infty)$ is a nonnegative decreasing function and (34) holds, then for any small enough $\epsilon > 0$ and any sufficiently large $R > 1$ we have

$$(45) \quad \int_{M \setminus B_R} f(r(x)) V(x)^{-\beta + \epsilon} d\mu \leq C \int_{\frac{R}{2}}^{+\infty} f(r) r^{\alpha + C_0 \epsilon - 1} (\log r)^b dr$$

for some positive constant C , see [16, formula (2.19)]. Recalling that $R^\delta = e$ and using inequality (45) in (44), we obtain

$$\delta^{-p \frac{\gamma + \sigma}{\sigma - p + 1}} I_2 \leq C \int_{\frac{R}{2}}^{+\infty} r^{-p \frac{\sigma + \gamma}{\sigma - p + 1} (C_1 \delta + 1) + \alpha + C_0 \frac{\delta}{\sigma - p + 1} - 1} (\log r)^b dr$$

Now we define

$$(46) \quad s := p \frac{\sigma + \gamma}{\sigma - p + 1} (C_1 \delta + 1) - \alpha - C_0 \frac{\delta}{\sigma - p + 1} = \frac{\delta}{\sigma - p + 1} ((\sigma + \bar{\gamma} - \delta)p C_1 - (p + C_0))$$

and by our assumptions on C_1 and δ , see also (42), we have

$$(47) \quad 0 < \frac{\delta}{\sigma - p + 1} \leq s \leq p C_1 \frac{(\sigma + \bar{\gamma})}{\sigma - p + 1} \delta.$$

Thus

$$\delta^{-p \frac{\gamma + \sigma}{\sigma - p + 1}} I_2 \leq C \int_{\frac{R}{2}}^{+\infty} r^{-s} (\log r)^b \frac{dr}{r}$$

and the change of variable $\xi = s \log r$ yields

$$(48) \quad \delta^{-p \frac{\gamma + \sigma}{\sigma - p + 1}} I_2 \leq C s^{-b-1} \int_1^{+\infty} e^{-\xi} \xi^b d\xi \leq C \delta^{-b-1}.$$

Now we insert inequalities (43) and (48) into (38) and we obtain

$$(49) \quad \int_{B_R} V |u|^{\sigma + \gamma} d\mu \leq \int_M V |u|^{\sigma + \gamma} \psi_n^{pk} d\mu \leq C [\delta^{-p \frac{\bar{\gamma} + \sigma - \delta}{\sigma - p + 1}} n^{-\frac{\delta}{\sigma - p + 1}} (\log(2nR))^b + \delta^{-b-1}].$$

Passing to the \liminf as n tends to infinity in the previous inequality we deduce

$$\int_{B_R} V |u|^{\sigma + \bar{\gamma} - \frac{1}{\log R}} d\mu \leq C (\log R)^{b+1}.$$

Then, passing to the \liminf as R tends to infinity and using Fatou's Lemma, we obtain

$$\int_M V |u|^{\sigma+\bar{\gamma}} d\mu = 0,$$

which immediately yields $u = 0$ a.e. in M , since $V > 0$ a.e. in M .

We now turn to the case when **(HP1)** holds, i.e. when we have (34) with $b = -1$. We use again Proposition 3.1, with the test functions $\psi_n = \varphi\eta_n$ as in (35). From formulas (43), (48) and (49) in the proof of the previous case, but now with $b = -1$ instead of $b < -1$, for every $R > 0$ large enough and every $n \in \mathbb{N}$ we have

$$(50) \quad \int_{B_R} V |u|^{\sigma+\bar{\gamma}} d\mu \leq C [\delta^{-p \frac{\bar{\gamma}+\sigma-\delta}{\sigma-p+1}} n^{-\frac{\delta}{\sigma-p+1}} (\log(2nR))^{-1} + 1].$$

Passing to the \liminf as n and R tend to infinity in (50), due to Fatou's Lemma we obtain

$$(51) \quad \int_M V |u|^{\sigma+\bar{\gamma}} d\mu \leq C.$$

Now we use Proposition 3.2 with the family of test functions $\psi_n = \varphi\eta_n$, and from (21) we deduce

$$(52) \quad \int_M V(x) |u|^{\sigma+\bar{\gamma}} \psi_n^{pk} d\mu \leq C \left[\int_{M \setminus B_R} V(x) |u|^{\sigma+\bar{\gamma}} d\mu \right]^{\frac{\gamma+p-1}{\sigma+\bar{\gamma}}} \left[\delta^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} (I_3 + I_4) \right]^{\frac{\sigma-p+1+\delta}{\sigma+\bar{\gamma}}}$$

where we set

$$\begin{aligned} I_3 &= \int_M V^{-\frac{\gamma+p-1}{\sigma-p+1+\delta}} \varphi^{p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} |\nabla \eta_n|^{p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} d\mu, \\ I_4 &= \int_M V^{-\frac{\gamma+p-1}{\sigma-p+1+\delta}} \eta_n^{p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} |\nabla \varphi|^{p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} d\mu. \end{aligned}$$

Now we proceed to estimate I_3, I_4 as I_1 and I_2 , respectively. Indeed, we start noting that

$$-\frac{\gamma+p-1}{\sigma-p+1+\delta} = -\beta + \frac{\sigma+\bar{\gamma}}{(\sigma-p+1)(\sigma-p+1+\delta)} \delta.$$

Thus we easily obtain

$$\delta^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} I_3 \leq \delta^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} n^{-p C_1 \delta \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} (nR)^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} \int_{B_{2nR} \setminus B_{nR}} V^{-\beta + \frac{\sigma+\bar{\gamma}}{(\sigma-p+1)(\sigma-p+1+\delta)} \delta} d\mu.$$

By condition (11) for $\delta > 0$ small enough we have

$$\delta^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} I_3 \leq C \delta^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} n^{\frac{(\sigma+\bar{\gamma})(C_0+p-C_1p(\sigma-p+1))}{(\sigma-p+1)(\sigma-p+1+\delta)} \delta} R^{\frac{(\sigma+\bar{\gamma})(p+C_0)}{(\sigma-p+1)(\sigma-p+1+\delta)} \delta} (\log(2nR))^{-1}.$$

Choosing $\delta > 0$ small enough and

$$(53) \quad C_1 > \frac{2+p+C_0}{(\sigma-p+1)p}$$

we obtain

$$(54) \quad \delta^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} I_3 \leq C \delta^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} n^{-\frac{\sigma+\bar{\gamma}}{(\sigma-p+1)^2} \delta} (\log(2nR))^{-1}.$$

On the other hand we have

$$\delta^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} I_4 \leq C R^{C_1 p \delta \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} \int_{M \setminus B_R} V^{-\beta + \frac{\sigma+\bar{\gamma}}{(\sigma-p+1)(\sigma-p+1+\delta)} \delta} r(x)^{-(C_1 \delta + 1)p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} d\mu.$$

By inequality (45) we obtain

$$(55) \quad \delta^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} I_4 \leq C \int_{\frac{R}{2}}^{+\infty} r^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta} (C_1\delta+1)+\alpha+C_0\delta \frac{\sigma+\bar{\gamma}}{(\sigma-p+1)(\sigma-p+1+\delta)}} (\log r)^{-1} dr,$$

and we set

$$t := p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta} (C_1\delta+1) - \alpha - C_0\delta \frac{\sigma+\bar{\gamma}}{(\sigma-p+1)(\sigma-p+1+\delta)}.$$

By (53) and for $\delta > 0$ small enough we have

$$0 < \frac{\sigma+\bar{\gamma}}{(\sigma-p+1)^2} \delta < t < 2 \frac{\sigma+\bar{\gamma}}{(\sigma-p+1)^2} \delta$$

and using the change of variables $\xi = t \log r$ in (55) we obtain

$$(56) \quad \delta^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} I_4 \leq C \int_1^{+\infty} e^{-\xi} \xi^{-1} d\xi \leq C.$$

Hence from (52), (54) and (56) we deduce that for every $n \in \mathbb{N}$ and every $R > 0$ large enough

$$\begin{aligned} \int_{B_R} V |u|^{\sigma+\bar{\gamma}-\delta} d\mu &\leq \int_M V |u|^{\sigma+\bar{\gamma}} \psi_n^{pk} d\mu \\ &\leq C \left[\int_{M \setminus B_R} V(x) |u|^{\sigma+\bar{\gamma}} d\mu \right]^{\frac{\gamma+p-1}{\sigma+\bar{\gamma}}} \left[\delta^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1+\delta}} n^{-\frac{\sigma+\bar{\gamma}}{(\sigma-p+1)^2} \delta} (\log(2nR))^{-1} + 1 \right]^{\frac{\sigma-p+1+\delta}{\sigma+\bar{\gamma}}}. \end{aligned}$$

Passing to the \liminf as n tends to infinity we have

$$\int_{B_R} V |u|^{\sigma+\bar{\gamma}-\delta} d\mu \leq C \left[\int_{M \setminus B_R} V(x) |u|^{\sigma+\bar{\gamma}} d\mu \right]^{\frac{\bar{\gamma}+p-1-\delta}{\sigma+\bar{\gamma}}}$$

and taking the \liminf as R tends to infinity, using (51) and Fatou's Lemma we finally infer that

$$\int_M V |u|^{\sigma+\bar{\gamma}} d\mu = 0,$$

and thus $u = 0$ a.e. in M . \square

Proof of Theorem 2.6. The proof of Theorem 2.6 follows along the same lines of that of Theorem 2.5 in the simpler case when $b < -1$. Indeed, we apply Proposition 3.1 with the same choice of test functions $\psi_n = \varphi \eta_n$ introduced in (35), and we obtain as in (38)

$$(57) \quad \int_M V |u|^{\sigma+\bar{\gamma}} \psi_n^{pk} d\mu \leq C \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} [I_1 + I_2],$$

with I_1, I_2 defined in (39). As in (40) we deduce

$$\delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} I_1 \leq \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} n^{-p C_1 \delta \frac{\sigma+\bar{\gamma}}{\sigma-p+1}} (nR)^{-p \frac{\sigma+\bar{\gamma}}{\sigma-p+1}} \int_{B_{2nR} \setminus B_{nR}} V^{-\frac{\bar{\gamma}+p-1-\delta}{\sigma-p+1}} d\mu.$$

Now by (12) we have, similarly to (43),

$$(58) \quad \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} I_1 \leq C \delta^{-p \frac{\bar{\gamma}+\sigma-\delta}{\sigma-p+1}} n^{-\frac{\delta}{\sigma-p+1}} (\log(2nR))^b e^{-\frac{\theta\delta}{\sigma-p+1} \log(nR) \log \log(nR)}.$$

In order to estimate I_2 , we note that as in (44) one has

$$(59) \quad \delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} I_2 \leq C R^{C_1 p \delta \frac{\sigma+\bar{\gamma}}{\sigma-p+1}} \int_{M \setminus B_R} V^{-\frac{\bar{\gamma}+p-1-\delta}{\sigma-p+1}} r(x)^{-(C_1\delta+1)p \frac{\sigma+\bar{\gamma}}{\sigma-p+1}} d\mu.$$

We now claim that if $f : [0, \infty) \rightarrow [0, \infty)$ is a nonnegative decreasing function and (12) holds, then for any small enough $\epsilon > 0$ and any sufficiently large $R > 0$ we have

$$(60) \quad \int_{M \setminus B_R} f(r(x)) V(x)^{-\beta+\epsilon} d\mu \leq C \int_{\frac{R}{2}}^{+\infty} f(r) r^{\alpha+C_0\epsilon-1} (\log r)^b e^{-\epsilon\theta \log \frac{r}{2} \log \log \frac{r}{2}} dr$$

for some positive fixed constant C . The proof of the claim is similar to that of [16, formula (2.19)] and of [18, formula (3.25)]. We sketch it here for the reader's convenience. By the monotonicity of the involved functions, using condition (12) we obtain

$$\begin{aligned} & \int_{M \setminus B_R} f(r(x)) V(x)^{-\beta+\epsilon} d\mu \\ &= \sum_{i=0}^{+\infty} \int_{B_{2^{i+1}R} \setminus B_{2^iR}} f(r(x)) V(x)^{-\beta+\epsilon} d\mu \\ &\leq \sum_{i=0}^{+\infty} f(2^i R) \int_{B_{2^{i+1}R} \setminus B_{2^iR}} V^{-\beta+\epsilon} d\mu \\ &\leq C \sum_{i=0}^{+\infty} f(2^i R) e^{-\epsilon\theta \log(2^i R) \log \log(2^i R)} (2^{i+1} R)^{\alpha+C_0\epsilon} (\log(2^{i+1} R))^b \\ &\leq \hat{C} \sum_{i=0}^{+\infty} f(2^i R) e^{-\epsilon\theta \log(2^{i-1} R) \log \log(2^{i-1} R)} (2^{i-1} R)^{\alpha+C_0\epsilon} (\log(2^{i-1} R))^b \\ &\leq \hat{C} \sum_{i=0}^{+\infty} \int_{2^{i-1}R}^{2^iR} f(r) e^{-\epsilon\theta \log \frac{r}{2} \log \log \frac{r}{2}} r^{\alpha+C_0\epsilon-1} (\log r)^b dr \\ &= \hat{C} \int_{\frac{R}{2}}^{+\infty} f(r) e^{-\epsilon\theta \log \frac{r}{2} \log \log \frac{r}{2}} r^{\alpha+C_0\epsilon-1} (\log r)^b dr. \end{aligned}$$

Now from (59) and (60) it follows that

$$\delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} I_2 \leq C \int_{\frac{R}{2}}^{+\infty} r^{-p \frac{\sigma+\gamma}{\sigma-p+1} (C_1\delta+1)+\alpha+\frac{\delta C_0}{\sigma-p+1}-1} (\log r)^b e^{-\frac{\delta\theta}{\sigma-p+1} \log \frac{r}{2} \log \log \frac{r}{2}} dr.$$

We define s as in (46), so that from the above inequality we obtain

$$\delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} I_2 \leq C \int_{\frac{R}{2}}^{+\infty} r^{-s} (\log r)^b e^{-\frac{\delta\theta}{\sigma-p+1} \log \frac{r}{2} \log \log \frac{r}{2}} \frac{dr}{r}.$$

The change of variable $\xi = s \log \frac{r}{2}$ then yields

$$\delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} I_2 \leq C 2^{-s} s^{-1} \int_{s \log \frac{R}{4}}^{+\infty} e^{-\xi} \left(\log 2 + \frac{\xi}{s} \right)^b e^{-\frac{\delta\theta}{s(\sigma-p+1)} \xi \log \frac{\xi}{s}} d\xi.$$

By inequality (47) we have

$$\delta^{-p \frac{\gamma+\sigma}{\sigma-p+1}} I_2 \leq C s^{-b-1} \left(\int_{s \log \frac{R}{4}}^{+\infty} e^{-\xi} \xi^b d\xi \right) e^{-\frac{\delta\theta}{\sigma-p+1} \log \frac{R}{4} \log \log \frac{R}{4}}$$

Note that by (47) we also have

$$s \log \frac{R}{4} \geq \frac{\delta}{\sigma-p+1} \log \frac{R}{4} = \frac{1}{\sigma-p+1} \frac{\log \frac{R}{4}}{\log R} \geq \frac{1}{2(\sigma-p+1)} > 0$$

for $R > 0$ large enough. Hence, using again (47), we deduce that

$$(61) \quad \begin{aligned} \delta^{-p\frac{\gamma+\sigma}{\sigma-p+1}} I_2 &\leq C\delta^{-b-1} \left(\int_{\frac{1}{2(\sigma-p+1)}}^{+\infty} e^{-\xi} \xi^b d\xi \right) e^{-\frac{\delta\theta}{\sigma-p+1} \log \frac{R}{4} \log \log \frac{R}{4}} \\ &\leq C\delta^{-b-1} e^{-\frac{\delta\theta}{\sigma-p+1} (\frac{1}{\delta} - \log 4) \log (\frac{1}{\delta} - \log 4)} = C\delta^{-b-1} e^{(-\frac{\theta}{\sigma-p+1} + \frac{\theta\delta \log 4}{\sigma-p+1}) \log (\frac{1}{\delta} - \log 4)}. \end{aligned}$$

By inequalities (57), (58) and (61), for $R > 0$ large enough and every $n \in \mathbb{N}$ we have

$$\begin{aligned} \int_{B_R} V |u|^{\sigma+\gamma} d\mu &\leq \int_M V |u|^{\sigma+\gamma} \psi_n^{pk} d\mu \\ &\leq C \left[\delta^{-p\frac{\gamma+\sigma-\delta}{\sigma-p+1}} n^{-\frac{\delta}{\sigma-p+1}} (\log(2nR))^b e^{-\frac{\theta\delta}{\sigma-p+1} \log(nR) \log \log(nR)} \right. \\ &\quad \left. + \delta^{-b-1} e^{(-\frac{\theta}{\sigma-p+1} + \frac{\theta\delta \log 4}{\sigma-p+1}) \log (\frac{1}{\delta} - \log 4)} \right], \end{aligned}$$

thus passing to the \liminf as n tends to infinity we obtain

$$(62) \quad \int_{B_R} V |u|^{\sigma+\gamma} d\mu \leq C\delta^{-b-1} e^{(-\frac{\theta}{\sigma-p+1} + \frac{\theta\delta \log 4}{\sigma-p+1}) \log (\frac{1}{\delta} - \log 4)}.$$

Since by our assumption $b+1 - \frac{\theta}{\sigma-p+1} < 0$, the function on the right-hand side in the above inequality tends to 0 as δ tends to 0^+ , hence passing to the \liminf as R tends to infinity in (62) and using Fatou's Lemma we conclude that

$$\int_M V |u|^{\sigma+\gamma} d\mu = 0,$$

and thus $u = 0$ a.e. in M . □

REFERENCES

- [1] M.F. Bidaut-Véron, Local and global behavior of solutions of quasilinear equations of Emden-Fowler type. *Arch. Rational Mech. Anal.*, 107(4):293–324, 1989.
- [2] M.F. Bidaut-Véron, L. Véron, Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations. *Invent. Math.*, 106(3):489–539, 1991.
- [3] D. Castorina, P. Esposito, B. Sciunzi, Degenerate elliptic equations with singular nonlinearities. *Preprint*.
- [4] L. Damascelli, B. Sciunzi, Regularity, monotonicity and symmetry of positive solutions of m -Laplace equations. *J. Differential Equations*, 206(2):483–515, 2004.
- [5] L. Damascelli, B. Sciunzi, Harnack inequalities, maximum and comparison principles, and regularity of positive solutions of m -Laplace equations. *Calc. Var. Partial Differential Equations*, 25(2):139–159, 2006.
- [6] L. Damascelli, A. Farina, B. Sciunzi, E. Valdinoci, Liouville results for m -Laplace equations of Lane-Emden-Fowler type *Ann. Inst. H. Poincaré, An. Non Lin.*, 26: 1099–1119, 2009.
- [7] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.*, 7(8) : 827 – 850, 1983.
- [8] L. D'Ambrosio, V. Mitidieri, A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities, *Adv. Math.*, 224:967–1020, 2010.
- [9] R.H. Fowler, Further studies of Emden's and similar equations. *Q.J. Math. Oxf. ser 2*, 2:259–288, 1931.
- [10] A. Farina, Liouville-type results for solutions of $-\Delta u = |u|^{p-1}u$ on unbounded domains of \mathbb{R}^N . *C. R. Math. Acad. Sci. Paris*, 341(7):415–418, 2005.
- [11] A. Farina, On the classification of solutions of the Lane-Emden equation on unbounded domains of \mathbb{R}^N . *J. Math. Pures Appl.*, 87(5):537–561, 2007.
- [12] A. Farina, B. Sciunzi and E. Valdinoci, Bernstein and De Giorgi type problems: new results via a geometric approach *Preprint*, mp_arc 07-171, 2007.

- [13] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 34(4):525–598, 1981.
- [14] B. Gidas and J. Spruck, Global A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations*, 6(8):883–901, 1981.
- [15] A. Grigor'yan, V. A. Kondratiev, On the existence of positive solutions of semilinear elliptic inequalities on Riemannian manifolds, *In Around the research of Vladimir Maz'ya. II, volume 12 of Int. Math. Ser. (N. Y.)*, pages 203–218. Springer, New York, 2010.
- [16] A. Grigor'yan, Y. Sun, On non-negative solutions of the inequality $\Delta u + u^\sigma \leq 0$ on Riemannian manifolds. *Comm. Pure Appl. Math.*, 67(8):1336–1352, 2014.
- [17] D.D. Joseph and T.S. Lundgren, Quasilinear Dirichlet problems driven by positive sources. *Arch. Rational Mech. Anal.*, 49(1-2):241–269, 1972/73.
- [18] P. Mastrolia, D.D. Monticelli and F. Punzo, Nonexistence results for elliptic differential inequalities with a potential on Riemannian manifolds. *Calc. Var. Partial. Diff. Eq.*, 54:1345–1372, 2015.
- [19] P. Mastrolia, D.D. Monticelli, F. Punzo, Nonexistence of solutions to parabolic differential inequalities with a potential on Riemannian manifolds, *Math. Ann.* (to appear), doi 10.1007/s00208-016-1393-2
- [20] V. Mitidieri, S. I. Pohozev, Absence of global positive solutions of quasilinear elliptic inequalities, *Dokl. Akad. Nauk*, 359: 456–460, 1998.
- [21] V. Mitidieri, S.I. Pohozaev, Nonexistence of positive solutions for quasilinear elliptic problems in \mathbb{R}^N , *Tr. Mat. Inst. Steklova*, 227: 192–222, 1999.
- [22] V. Mitidieri, S.I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, *Tr. Mat. Inst. Steklova*, 234: 1–384, 2001.
- [23] V. Mitidieri, S.I. Pohozaev, Towards a unified approach to nonexistence of solutions for a class of differential inequalities, *Milan J. Math.*, 72: 129–162, 2004.
- [24] S.I. Pohozaev, A. Tesei, Nonexistence of local solutions to semilinear partial differential inequalities. *Ann. Inst. H. Poinc. Anal. Non Lin.*, 21:487–502, 2004.
- [25] F. Punzo, A. Tesei, On a semilinear parabolic equation with inverse-square potential, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 21: 359–396, 2010.
- [26] Y. Sun, Uniqueness results for nonnegative solutions of semilinear inequalities on Riemannian manifolds *J. Math. Anal. Appl.* 419: 646–661, 2014.
- [27] Y. Sun, On nonexistence of positive solutions of quasilinear inequality on Riemannian manifolds, *preprint (2013)*, <https://www.math.uni-bielefeld.de/sfb701/files/preprints/sfb13068.pdf>.